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Research Article

Prey-Predator Model for Awash National Park, Oromia, Ethiopia and Its Stability Analysis with Simulations

Srinivasarao Thota

Department of Applied Mathematics, School of Applied Natural Sciences, Adama Science and Technology University, Post Box No. 1888, Adama, Ethiopia
E-mail: srinivasarao.thota@astu.edu.et

Abstract

In this paper, we discuss a model of two-space food chain consisting of the population of ecology of foxes (the predator) and rabbits (prey) in Awash National park, Ethiopia. The study is based on formulation of a mathematical model to study the dynamics of the population densities and analyzing the stability of equilibrium points of the prey-predator model. The aim of this model is to explore the behavior of a simple model by considering a population of foxes, and rabbits. This model is constituted by a system of nonlinear decoupled ordinary differential equations. By using perturbed method, we identify the nature of the system at each equilibrium point. The stability analysis of a prey-predator model is discussed with numerical simulations.

Keywords: Pre-predator model, Stability Analysis, Equilibrium points, Non-linear differential equations, Numerical simulations.

Introduction

Mathematical population models have been used to study the dynamics of prey-predator systems since Lotka and Volterra proposed the simple model (Lotka, 1925) of prey-predator interactions, now called the Lotka-Volterra model [LVE]. The Lotka–Volterra predator–prey model was initially proposed by Alfred. Lotka in the theory of autocatalytic chemical reactions in 1910 (Lotka, 1910; Goel et al., 1971). This was effectively the logistic equation (Berryman, 1992) originally derived by Pierre François Verhulst (Verhulst, 1838). In 1920 Lotka extended the model, via Andrey Kolmogorov, to “organic systems” using a plant species and a herbivorous animal species and in 1925 he used the equations to analyse predator-prey interactions in his book on biomathematics (Lotka, 1925). The same set of equations was published in 1926 by Vito Volterra, a mathematician and physicist, who had become interested in mathematical biology (Goel et al., 1971); (Volterra, 1926); (Volterra, 1931). In the late 1980s, an alternative to the Lotka–Volterra predator–prey model (and its common-prey-dependent generalizations) emerged, the ratio dependent or Arditi–Ginzburg model (Arditi and Ginzburg, 1989). The validity of prey- or ratio-dependent models has been much debated [Abrams and Ginzburg, 2000). The Lotka–Volterra equations have a long history of use in economic theory; their initial application is commonly credited to Richard Goodwin in 1965 (Gandolfo, 2008) or 1967 (Goodwin, 1967), (Desai and Ormerod, 1998). Since Lotka-Volterra model proposed, many mathematical models have been constructed based on more realistic explicit and implicit biological assumptions, see, for example (Dennis et al, 2009; Genry, 2007; Jha et al., 2017; Kar, 2010; Lotka, 1925; Pulley et al., 2011; Suresh et al., 2017; Sunitha et al., 2016; Taleb, 2013; Volterra, 1926). All these mathematical models constituted by non-linear differential systems. Hence the system of non-linear differential equations plays a central role in modeling population dynamics in ecology. We formulate and study a model involving prey-predator's systems.

Research was performed from late April through June 2019 at Adama Science and Technology University (ASTU) located in Adama, Ethiopia. On the interactions of prey-predator animals, data are collected from internet and Awash National Park (ANP). ANP is located in the Rift Valley of Ethiopia, between $08^0 45'$N and $09^0 15'$N latitude and $39^0 40'$E and $40^0 10'$E longitude. The total area is about 756 km^2, which is bordered by the Awash River on the south and northeast, and Awash west wildlife reserve on the north and west.

Physical Meaning

The Lotka–Volterra model makes a number of assumptions, not necessarily realizable in nature, about the environment and evolution of the predator and prey populations (PPD, 2018):

- The prey population finds ample food at all times.
- The food supply of the predator population depends entirely on the size of the prey population.
- The rate of change of population is proportional to its size.
- During the process, the environment does not change in favour of one species, and genetic adaptation is inconsequential.
- Predators have limitless appetite.

As differential equations are used, the solution is deterministic and continuous. This, in turn, implies that the generations of both the predator and prey are continually overlapping (Cooke et al., 1981).

Figure 1: The prey-predator interaction of foxes and rabbits.

Materials and Methods

Let x and y be the population of the foxes and rabbits at any time t. The main feature of the model that two different functional responses of the predator are incorporated in the model to represent the difference in the way the predator feeds on each of the prey species. Terms representing logistic growth of the prey species in absence of the predator are included in the prey equations. Interspecific competition among the prey species is also included in the model. The model has two non-linear autonomous ordinary differential equations describing how the population densities of the two species would vary with time. Before construction of the model, we have the following assumption.

Assumptions for Model

The following assumptions are made in order to construct the model.

(i). Prey birth rate is proportional to the size of the population.
(ii). Predator birth rate is proportional to the size of both the predator and prey population.
(iii). Prey death rate is proportional to the size of both the predator and prey population.
(iv). Predator death rate is proportional to the size of the predator population.
(v). The species live in an ecosystem where external factors such as droughts, fires,

epidemics are stable or have a similar effect on the interacting species.

(vi). One prey is easy to capture by the predator, while the other prey has adopted to capture it.

(vii). There is logistic growth of the prey in absence of the predator or human poaching of the prey. That is the population of the prey would increase (or decrease) exponentially until it reaches the maximum density of the park.

(viii). The rate of increase of the predator population depends on the amount of prey biomass it converts as food.

Mathematical Model

Suppose the non-dimensional population density of the prey is *x* at time *t*, the population density of the predator is *y* at time *t*. In order to formulate the model, we have to use the above assumption with proportionality.

(i). The growth rate of any species at a given time is proportional to the number of species present at that time.

(ii). The species are living in a homogenous environment and age structures are not taken into consideration.

(iii). In the absence of the predators, the prey population would grow at logistic growth (natural rate), say α with $\dfrac{dx}{dt} = \alpha x,\ \alpha > 0$.

(iv). In the absence of the prey, the predator population would decline at natural rate, say β with $\dfrac{dy}{dt} = \beta y,\ \beta > 0$.

(v). When both prey and predator are present, the specific growth rate of prey is diminished by an amount population to the predator population and the growth rate of population enhanced proportional to prey population, consequently the effect of predator eating prey is an interaction rate of decline $-\gamma xy$ in the prey population *x* and an interaction rate of growth $-\gamma xy$ of the predator population *y* with δ and γ positive constants.

Under the above assumption proportionality, we modeled the prey-predator equations.

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - \gamma xy, \\ \frac{dy}{dt} &= \delta xy - \beta y, \end{aligned} \tag{1}$$

and $x(0) \geq 0,\ y(0) \geq 0$, where the parameter α is the prey birth (growth) rate, β is the predator death rate, δ and γ are the measure of the effect of the interaction between the two species of predator and prey respectively.

The Lotka-Volterra model makes a number of assumptions, not necessarily realizable in nature, about the environment and evolution of the predator and prey population. (i) The prey population finds ample food at all times. (ii) The food supply of the predator population depends on the size of the prey population. (iii) The rate of change of population is proportional to its size. (iv) During the process, the environment does not change unfavorable to one species, and genetic adaptation is inconsequential. (v) Predator has limitless appetites.

Although the Lotka-Volterra model is the best model currently available to accurately portray mathematically the population variation dynamics of a predator relationship, there are still several holes in science. All of the assumptions that go into this kind of model development naturally inflict some “holes” at the same time. Restrictions and limitations of this research exist in the assumptions it is built on. In my view, the greatest limitation of this model has against it is its lack of relevance to the majority of predation relationships. In order for this model to accurately portray the reality of population variance, the two species involved must satisfy (or close to satisfy) those major pieces to the puzzle. The model works for different animals, but the model not express explicitly for some predator prey interaction. The model is limited to predator relationships that rely explicitly on one another, free of the variables of climate change, natural disaster, hunting, or alternative food supplies.

We can also plot the solutions of the model equations (1) parametrically as orbits in phase space, without representing time, but with one axis representing the number of rabbits (prey) and the other axis representing the number of foxes (predator) for all times. In the model equation (1), we can eliminate time to produce a single differential equation in x and y as follows

$$\frac{dy}{dx}=\frac{\delta xy-\beta y}{\alpha x-\gamma xy}. \quad (2)$$

Applying variable separable technique to the differential equation (2), one can get the implicit relationship. Indeed, we have

$$\frac{dy}{dx}=\frac{y}{x}\frac{(\delta x-\beta)}{(\alpha-\gamma y)}\Rightarrow\frac{(\alpha-\gamma y)}{y}dy=\frac{(\delta x-\beta)}{x}dx$$
$$\Rightarrow\left(\frac{\alpha}{y}-\gamma\right)dy=\left(\delta-\frac{\beta}{x}\right)dx$$
$$\Rightarrow\alpha\ln y-\delta y+C=\delta x-\beta\ln x$$
$$\Rightarrow C=\delta x-\beta\ln x-\alpha\ln y+\delta y.$$

The solutions of the equation (2) are closed curves, where C is a constant quantity depending on the initial conditions and conserved on each curve.

Stability Analysis

In this paper, the Jacobin matrix method is used to linearized the non-linear system of prey-predator equations and also different MATLAB commends are used for solving differential equations (like ode45, ode23, etc.) to test the stability analysis of the model under different parameters.

Equilibrium Points

The equilibrium points of the system are necessary for the purpose of studying the local stability nature of the ecological model (1). The system, under investigation, has the following two equilibrium points.

(i) Fully washed state or extent state: $E_1=(0,0)$.

(ii) Coexistence state $E_2=\left(\frac{\beta}{\delta},\frac{\alpha}{\gamma}\right)$.

Existence and Stability Analysis of Equilibrium Points

The Jacobin matrix for the system (1) at equilibrium point $E=(x,y)$ is given by

$$J_E=\begin{bmatrix}\alpha-\lambda y & -\gamma x\\ y\delta & -\beta+x\delta\end{bmatrix} \quad (3)$$

Theorem 1: The system is always exists and unstable at the equilibrium points E_1.

Proof: We have $E_1=(0,0)$, it is clear that the equilibrium point E_1 exists and the corresponding Jacobin matrix at E_1 is

$$J_{E_1}=\begin{bmatrix}\alpha & 0\\ 0 & -\beta\end{bmatrix}$$

From J_{E_1}, we have the 9values $\lambda_1=-\beta<0$ and $\lambda_2=\alpha>0$. Hence the equilibrium point is saddle point and the dynamical system (1) is unstable.

Theorem 2: The interior equilibrium point $E_2=\left(\frac{\beta}{\delta},\frac{\alpha}{\gamma}\right)$ is always neutrally stable.

Proof: We have $E_2=\left(\frac{\beta}{\delta},\frac{\alpha}{\gamma}\right)$, where $\frac{\beta}{\delta}>0$ and $\frac{\alpha}{\gamma}>0$. Hence the equilibrium point E_2 exists and the corresponding Jacobin matrix at E_2 is

$$J_{E_2} = \begin{bmatrix} 0 & -\dfrac{\beta\gamma}{\delta} \\ \dfrac{\alpha\delta}{\gamma} & 0 \end{bmatrix}$$

The eigenvalues of J_{E_2} are $\lambda_1 = i\sqrt{\alpha\beta}$ and $\lambda_2 = -i\sqrt{\alpha\beta}$. Hence the dynamical system (1) at the equilibrium point E_2 is always neutrally stable.

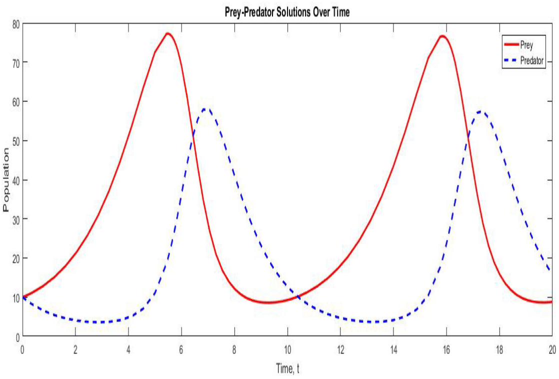
Results and Discussions

In this section, the systems of equations given in (1) are solved numerically using the MATLAB software by fixing some of the parameters values in the model. From the graphical representations (Figure 2, Figure 3), one can identify how the system is changing its behavior.

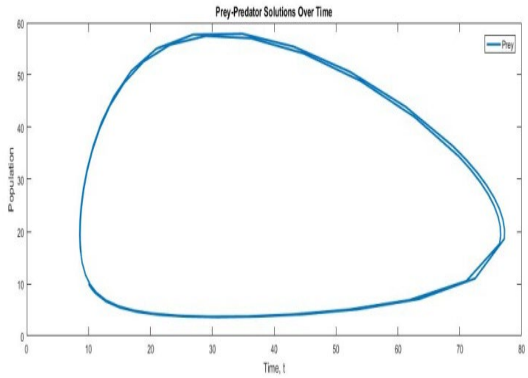
Indeed, when both prey and predator are present, the specific growth rate of prey is diminished by an amount population to the predator population and the growth rate of population enhanced proportional to prey population, consequently the effect of predator eating prey is an interaction rate of decline γxy in the prey population x and an interaction rate of growth γxy of the predator population y with δ and γ positive constants. In particular, by selecting the parameter values (assume x and y quantify thousands each) given below in the dynamical system (1),

$$\alpha = 0.55, \beta = 0.84, \gamma = 0.028, \delta = 0.027$$

with initial conditions $x(0) = y(0) = 10$, we have the corresponding eigenvalues $\lambda_1 = 0.6797058187i$, $\lambda_1 = -0.6797058187i$ and equilibrium point $E_2 = (31.11111111, 19.64285714)$. The equations (1) describe predator and prey population dynamics in the presence of one another, and together make up the Lotka-Volterra predator-prey model. The model predicts a cyclical relationship between predator and prey numbers: as the number of predators (y) increases so does the consumption rate (γxy), tending to reinforce the increase in y. Increase in consumption rate, however, has an obvious consequence - a decrease in the number of prey (x), which in turn causes y (and therefore γxy) to decrease. As γxy decreases the prey population is able to recover, and x increases. Now y can increase, and the cycle begins again. This graph in Figure 2 shows the cyclical relationship predicted by the model for hypothetical predator and prey populations. Therefore, the system (1) is always neutrally stable by observing the species growth rates.



(a)



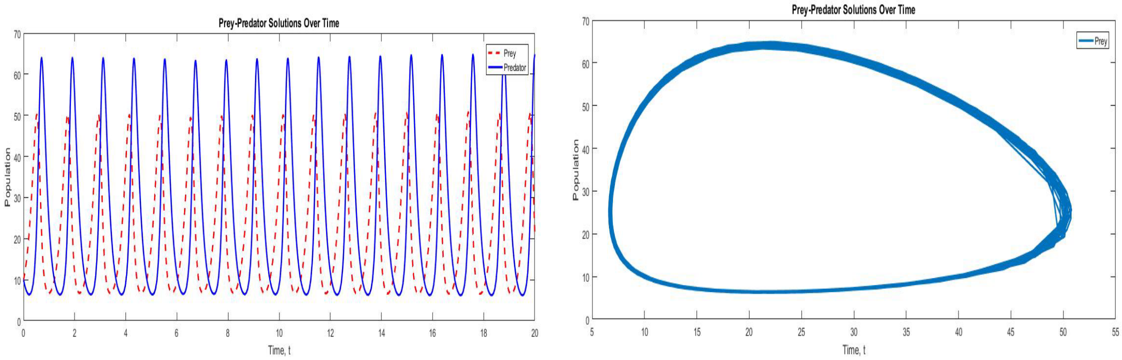
(b)

Figure 2 $\alpha = 0.55, \beta = 0.84, \gamma = 0.028, \delta = 0.027$ for $E_2 = (31.11111111, 19.64285714)$

Now, taking the parameter values given below in the dynamical system (1),

$$\alpha = 0.5, \beta = 6.5, \gamma = 0.2, \delta = 0.3$$

with initial conditions $x(0) = y(0) = 10$, we get the corresponding eigenvalues $\lambda_1 = 5.700877125i$, $\lambda_1 = -5.700877125i$ and the equilibrium point $E_2 = (21.66666667, 25.00000000)$.



(a) **(b)**

Figure 3: $\alpha = 0.5, \beta = 6.5, \gamma = 0.2, \delta = 0.3$ for $E_2 = (21.66666667, 25.00000000)$

Following the discussion given for Figure 2, we can observe that the prey population increases when there are no predators, but the predator population decreases when there are no prey. Therefore, we can conclude that the system (1) is always neutrally stable by observing the species growth rates in Figure 3.

MATLAB Code for Simulations

In this section, we provide MATLAB simulation code to obtain the graphical representation of the model. The main MATLAB code is as follows.

```
[t,y]=ode45(@prey_predator,[0 20],[30;4])
plot(t,y(:,1), 'red',t,y(:,2),'blue');    % for 2D over time
%plot(y(:,1),y(:,2))                      % for phase portrait
xlabel('Time, t');
ylabel('Population');
title('Prey-Predator Solutions Over Time');
legend('Prey','Predator')
```

The sub-code of MATLAB consisting of non-linear equations with constants is given as follows.

```
function myEqs = prey_predator (t, y)
a = 5.0; %0.55;
c = 0.2; %0.028;
b = 6.5; %0.84;
d = 0.3; %0.027;
myEqs = [a*y(1)-c*y(1)*y(2);
        -b*y(2)+d*y(1)*y(2)];
```

Conclusion

In this paper, we presented a model of two-space food chain consisting of the population of ecology of foxes (predator) and rabbits (prey) in Awash National park, Ethiopia. The aim of this paper is to explore the behavior of a simple model by considering a population of foxes and rabbits. This model is constituted by a system of non-linear decoupled first order ordinary differential equations. By using perturbed method, we investigate the local stability nature of the system at each possible equilibrium point. Further, the numerical illustrations at suitable parametric values to the model are presented by observing the species survivalness in nature for long time.

One can also extend the proposed model to three species model (for example, interaction between foxes, rabbits and rats), and following the procedure of the proposed model, we can give conclusion about the sustainability of foxes, rabbits and rats.

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References

Abrams, P. A.; Ginzburg, L. R. (2000). The nature of predation: prey dependent, ratio dependent or neither?. Trends in Ecology & Evolution. 15 (8): 337–341.

Arditi, R.; Ginzburg, L. R. (1989). Coupling in predator-prey dynamics: ratio dependence. Journal of Theoretical Biology. 139 (3): 311–326.

Berryman, A. A. (1992). The Origins and Evolution of Predator-Prey Theory. Ecology. 73 (5): 1530–1535.

Cooke, D.; Hiorns, R. W.; et al. (1981). The Mathematical Theory of the Dynamics of Biological Populations. II. Academic Press.

Dennis G. Zill., Michael R. (2009). Differential Equations with Boundary Value Problems, 7th edition.

Desai, M.; Ormerod, P. (1998). Richard Goodwin: A Short Appreciation. The Economic Journal. 108 (450): 1431–1435.

Gandolfo, G. (2008). Giuseppe Palomba and the Lotka–Volterra equations. Rendiconti Lincei. 19 (4): 347–357.

Genry J. Ricardo (2007). A Modern Introduction to Differential Equation. College City University of New York.

Goel, N. S.; et al. (1971). On the Volterra and Other Non-Linear Models of Interacting Populations. Academic Press.

Goodwin, R. M. (1967). A Growth Cycle. In Feinstein, C. H. (ed.). Socialism, Capitalism and Economic Growth. Cambridge University Press.

Jha P. K., Ghorai, S. (2017). Stability of Prey-Predator Model with Holling type Response Function and Selective Harvesting. J Appl Computat Math 6: 358.

Kar, T. K. (2010). Stability Analysis of A Prey-Predator Model Incorportaion a Prey Refuge. Commun. Nonlinear Sci. Numer. Simul. 10: 681-691.

Lotka, A. J. (1910). Contribution to the Theory of Periodic Reaction. J. Phys. Chem. 14 (3): 271–274.

Lotka, A. J. (1925). Elements of Physical Biology. Williams and Wilkins.

LVE . “Lotka–Volterra equations”. https://en.wikipedia.org/wiki/Lotka%E2%80%93Volterra_equations.

PPD (2018). “Predator-Prey Dynamics”. www.tiem.utk.edu. Retrieved 2018-01-09.

Pulley, Lucas, C. (2011). Analyzing Predator-Prey Models Using Systems of Ordinary Linear Differential Equations. Honors Thesis. 344.

Suresh K., Seshagiri Rao, N., Kalyani, K. (2017). An Ecological Model of Mutualism Interaction between Two Species and a Predator. International Journal of Mathematics & Computation. 28(4): 48-57.

Sunitha G, Komal G (2016). A three species dynamical system involving prey-predator, competition and commensalism. Applied Mathematics and Computation, 273: 54-67.

Taleb A. S. Obaid (2013). The Predator-Prey Model Simulation. Basrah Journal of Science. 31 (2): 103-109.

Verhulst, P. H. (1838). Notice sur la loi que la population poursuit dans son accroissement. Corresp. Mathématique et Physique. 10: 113-121.

Volterra, V (1926). Variazioni e fluttazionidelnumero di individui in species animaliconviventi. MemorieAccademiaNazionaledeiLincei. 2: 31-313.

Volterra, V. (1931). "Variations and fluctuations of the number of individuals in animal species living together". In Chapman, R. N. (ed.). Animal Ecology. McGraw–Hill.