

Implicative Ideal of a Pseudoring

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Abstract

In this study, we present the notion of an implicative ideal of a pseudoring and characterize it. We also introduce the notion of a finite product property of a subset of a pseudoring and prove its relations with the implicative and other types of ideal of a pseudoring. Moreover, we demonstrate the relationship between an implicative ideal with prime, polar ideals, as well as the atom of an atomic pseudoring. Finally, we prove that a prime ideal is implicative if and only if it is a maximal ideal of R .

Keywords: Finite product property, Implicative ideal of a pseudoring, Maximal ideal of a pseudoring, Prime ideal of a pseudoring.

Introduction

Implicative algebras offer a strong and flexible tool for grasping the fundamentals of logic and computation by establishing a common framework for different model constructs, for example, Miquel (2020) make it possible to simulate intuitionistic and classical logic while obfuscating the distinction between types and proofs. Rasiowa (1974) demonstrated how different kinds of implicative algebras can be represented by more tangible structures, such as Boolean algebras or their generalizations by proving representation theorems regarding these algebras. Chadja and Langer (2007) utilized, especially to non-classical logics and associated algebraic structures, to examine the algebraic features of the implication connective. Similarly, an implicative ideal has applications in fuzzy logic. Thus, the following results were some of the studies done in different algebras by the scholars.

In the theory of MV-algebras, implicative ideals were studied by different authors, for instance, Hoo and Sessa (1994) explored the

features of implicative ideals and made linkages between them and other sorts of ideals, including prime ideals and maximal ideals, in order to apply the idea of implicative ideals to their work on MV-algebras, particularly in the setting of pseudo MV-algebras. Hoo (1997) investigated implicative ideals in MV-algebras, emphasizing their characteristics and linkages to other kinds of ideals, as well as how they relate to linearly ordered ideals, essential ideals, and maximal ideals. Hoo (1989) examined implicative ideals in relation to MV-algebras, particularly how they are used to describe and categorize various MV-algebra types. Also it was studied in the theory of BL-algebra by Walendziak (2024).

Bandaru and Shum (2014) studied the idea of implicative ideals and the fuzzy implicative ideals of a distributive implication groupoid. A finite union property of non-empty subsets of MV-algebra was introduced by Saidi and Borzooei (2020). Chajda and L'anger (2004) also introduced pseudoring. Natei *et al.* (2023) studied ideals, extended and metric ideals of a pseudoring, and introduced the polar ideal and

atom of a pseudoring. However, as far as the knowledge of the researcher goes, the implicative ideal of a pseudoring and the finite product property of a nonempty subset of a pseudoring are not yet studied. In this paper, the authors introduce an implicative ideal and establish the necessary and sufficient requirements for an ideal of a pseudoring to be characterized as implicative. Also, investigate the finite product property of a subset of a pseudoring and prove its relations with the implicative ideal. Moreover, the researcher demonstrates the relationship between an implicative ideal and prime, polar ideals, as well as the atom of an atomic pseudoring. Finally, the researchers prove that a prime ideal is implicative if and only if it is a maximal ideal of R . Throughout this paper; R denotes a pseudoring, if not otherwise stated.

Chajda and Langer (2004) stated that, an algebra $R = (R, +, \cdot, 1)$ of type $(2,2,0)$ satisfying $(R, \cdot, 1)$ is commutative monoid, the property.

$$1 + (1 + x) = x, \quad x0 = 0, \quad (1 + x(1 + y))(1 + y) = (1 + y(1 + x))(1 + x)$$

and

$$1 + (1 + x(1 + y))(1 + y(1 + x)) = x + y$$

holds true is called a pseudoring, where 0 denotes the element $1 + 1$.

Remark 1.1. Following from the definition of R , it is clear that $+$ is commutative.

“ $x \leq y$ if and only if $x(y + 1) = 0$.” (Chajda and Langar, 2004).

Definition 1.1. “A subset I of R containing zero that satisfies the property, for $x, y \in I, 1 + (x + 1)(y + 1) \in I$ and for any $y \in I$ and $x \in R$, if $x \leq y$ then $x \in I$ is called an ideal of R ” (Natei *et al.*, 2023).

Proposition 1.1. “A subset I of R , which contain zero is an ideal if and only if for $x, y \in I \Rightarrow 1 + (x + 1)(y + 1) \in I$ and the last property of an ideal in the Definition 1.1 is replaced by the property,

$$(y + 1)x, y \in I \Rightarrow x \in I$$
 (Natei *et al.*, 2023).

Proposition 1.2. “For any $x, y \in R, y(1 + (x + 1)y) = x(1 + (y + 1)x)$ (Natei *et al.*, 2023).

Definition 1.2. “A homomorphism is a mapping $\varphi : R_1 \rightarrow R_2$ which preserves the multiplicative identity and homomorphism with respect to both binary operations” (Natei *et al.*, 2023).

Proposition 1.3. If R is a pseudoring, then the properties $x(x + 1) = 0, y(1 + 0) = y, \forall y \in R,$ and $x + 0 = x, Char(R) = 0$ hold on R (Natei *et al.*, 2025).

Proposition 1.4. The relation \leq preserves multiplication on R and $x \leq y \Rightarrow y + 1 \leq x + 1$ (Natei *et al.*, 2025)

Definition 1.3. For any $a \in R$, define $a/I = \{x \in R : x + a \in I\}$ and for any $a, b \in R, a/I = b/I$ if and only if $a + b \in I$, where I is an ideal of R (Natei *et al.*, 2025).

Notation: $a/I = \bar{a}$.

Theorem 1.1. “The set $R/I = \{x/I : x \in R\}$ with the operations $x/I + y/I = (x + y)/I = \{r \in R : r + (x + y) \in I\}, x/I \cdot y/I = (x \cdot y)/I$ is a pseudoring for any ideal I of R ” (Natei *et al.*, 2025).

Proposition 1.5. The map $\varphi_I : R \rightarrow R/I$ by $\varphi_I(x) = x/I$ is an onto homomorphism, where I is an ideal of R .

Definition 1.4. “A proper ideal P of R is called prime ideal if for every $x, y \in R$, either $x(y + 1) \in P$ or $y(x + 1) \in P$ ” (Natei *et al.*, 2023).

“The intersection of any family of ideals of R is an ideal. Let H be a subset of R . Then the intersection of all ideals $I \supseteq H$ is the smallest

ideal containing H and is denoted by $\langle H \rangle$ (Natei *et al.*, 2025).

Definition 1.5. The ideal generated by the set of all nil potent elements of R is called nil-radical of R , and denoted by $N = \langle Nil(R) \rangle$.

Definition 1.6. “Let M be a proper ideal of R , then M is called maximal if and only if $x \in M$ or $x + 1 \in M$ but not both for every $x \in R$ ” (Natei *et al.*, 2023).

Definition 1.7 “Let X be a non-empty subset of R . Then the set $X^\perp = \{y \in R : y(1 + (x + 1)y) = 0, \forall x \in X\}$ is called a polar of X in R ” (Natei *et al.*, 2025).

Definition 1.8. “A non-zero element h of R is called an atom if:

$$(h + 1)x = 0, \text{ then either } x = 0 \text{ or } x = h” \text{ (Natei } et al., 2025).$$

Theorem 1.2. “An atomic pseudoring R contains an atom $h \in M$ such that $h^\perp = M$ for any maximal ideal M of R ” (Natei *et al.*, 2025).

Theorem 1.3. “Let X is a non-empty subset of R . X^\perp is an ideal of R ” (Natei *et al.*, 2025).

Theorem 1.4. “If $X \subseteq R$, then the following are true.

- i. If $X \cap X^\perp \neq \emptyset$, then $X \cap X^\perp = \{0\}$.
- ii. $X \subseteq X^{\perp\perp}$ ” (Natei *et al.*, 2025).

Proposition 1.6. “For every non-zero ideal I of R , I^\perp is a proper subset of R ” (Natei *et al.*, 2025).?”

Implicative Ideals

Proposition 2.1. Let $\varphi : R_1 \rightarrow R_2$ is a pseudoring epimorphism. If I be an ideal of R_1 and $ker(\varphi) \subseteq I$, then $\varphi(I)$ is an ideal of R_2 .

Proof. Let $x \in R_2$ and $y \in \varphi(I)$ with

$x \leq y$. Since φ is onto $\exists a, b \in R_1$ such that $\varphi(a) = x$, $\varphi(b) = y \Rightarrow (y + 1)x = 0 \Rightarrow \varphi(a)(\varphi(b) + 1) = \varphi(0) = 0 \Rightarrow \varphi(a(b + 1)) = 0 \Rightarrow a(b + 1) \in ker(\varphi) \subseteq I \Rightarrow$
by Proposition 1.1, $a \in I$.

Lemma 2.1. Let H be a subset of R . The set $\langle H \rangle$ has following properties: If $H = \emptyset$, then $\langle H \rangle = \{0\}$. If $H \neq \emptyset$, then $\langle H \rangle = \{x \in R | 1 + \prod_{i=1}^n (h_i + 1)\}$ for some $h_i \in H$ and $n \in \mathbb{N}$.

Proof. Let H be a subset of R . If $H = \emptyset$, $I = \{0\}$ is clearly the smallest ideal that contains H . Let $H \neq \emptyset$ and let $T = \{x \in R : x \leq 1 + \prod_{i=1}^n (h_i + 1)\}$ for some $h_i \in H$.

We want to show that T is an ideal of R containing H . Let $a \in H$ we know that $a \leq a = 1 + (1 + a) \Rightarrow a \in T \Rightarrow H \subseteq T$.

(i) Clearly $0 \in T$.

(ii) Let $x, y \in T \Rightarrow x \leq 1 + \prod_{i=1}^n (h_i + 1)$ for some $h_i \in H$, and $i = 1, 2, \dots, n$ and $y \leq 1 + \prod_{j=1}^m (k_j + 1)$ for some $k_j \in H$. Let $h = 1 + \prod_{i=1}^n (h_i + 1)$ and $k = 1 + \prod_{j=1}^m (k_j + 1)$. It follows that $x \leq h$ and $y \leq k$. By Proposition 1.4,

$$h + 1 \leq x + 1 \text{ and } k + 1 \leq y + 1 \Rightarrow (h + 1)(k + 1) \leq (x + 1)(k + 1) \leq (x + 1)(y + 1) \Rightarrow 1 + (x + 1)(y + 1) \leq 1 + (h + 1)(k + 1).$$

Replacing h, k with their respective values

$$1 + (x + 1)(y + 1) \leq 1 + (\prod_{i=1}^n (h_i + 1) + 1)(\prod_{j=1}^m (k_j + 1)) \Rightarrow 1 + (x + 1)(y + 1) \in T.$$

(iii) Let $x \leq y, y \in T \Rightarrow y \leq 1 + \prod_{i=1}^n (h_i + 1)$ for some $h_i \in H$

$$\Rightarrow x \leq 1 + (h_1 + 1)(h_2 + 1) \cdots (h_n + 1) \Rightarrow x \in T$$

Therefore T is an ideal of R containing H . Let $x \in T \Rightarrow x \leq 1 + \prod_{i=1}^n (h_i + 1)$ for some $h_i \in H$. Assume that $n = 2$, then $x \leq 1 + \prod_{i=1}^2 (h_i + 1) \in \langle H \rangle$. Because $h_1, h_2 \in H \subseteq \langle H \rangle$ and $\langle H \rangle$ is an ideal. This implies $x \in \langle H \rangle$.

The same is true for any $n \in N$. Therefore, $T \subseteq \langle H \rangle$. Since $\langle H \rangle$ is the smallest ideal containing H , implies $T = \langle H \rangle$.

Definition 2.1. Let I represent R 's ideal. Then, I is referred to as an implicative ideal if any $x, y, z \in R$ with $x \cdot y \cdot z \in I$ and $x(y + 1) \in I \Rightarrow xz \in I$.

Theorem 2.1. Let I be an ideal of R . Then the following statements are equivalent.

- i. I is implicative.
- ii. For $x \in R$, with $x^n \in I \Rightarrow x \in I$ for $n \geq 1$.
- iii. For $x, y \in R$, with $x \cdot y^2 \in I \Rightarrow xy \in I$
- iv. For $x, y, z \in R$ with $x \cdot y \cdot (z + 1) \in I$ and $zx \in I \Rightarrow xy \in I$.

Proof. (i \Rightarrow ii) Suppose I is an implicative ideal of R . Let $x^n \in I \Rightarrow x \in I$. If $n = 1$, it is obvious. If $n = 2$, then $x^2 = x \cdot x \cdot 1 \in I$ and since I is an ideal, $x(x + 1) = 0 \in I$. Thus by Definition 2.1, $x \cdot 1 = x \in I$. Consider $x^n = x^{n-2} \cdot x \cdot x \in I$ and $x^{n-2} \cdot (x + 1) = 0 \in I$ for $n > 2$. Thus by Definition 2.1, $x^{n-2} \cdot x = x^{n-1} \in I$. Now again $x^{n-1} = x^{n-3} \cdot x \cdot x \in I$ and $x^{n-3} \cdot (x + 1) = 0 \in I$. Thus by Definition 2.1, $x^{n-2} \in I$. Continuing in this manner we find that $x^2 \in I$.

(ii \Rightarrow iii.) Suppose (ii) holds. Let $xy^2 \in I$. Since $x^2y^2 \leq xy^2 \in I \Rightarrow (xy)^2 \in I$. By

(ii) it follows that $xy \in I$.

(iii \Rightarrow iv). Suppose (iii) holds. Let $x \cdot y \cdot (z + 1) \in I$ and $zx \in I \Rightarrow x \cdot y \cdot t \in I$ and $(t + 1)x \in I$ for some $t = z + 1$. Since $x \leq 1 + (x(t + 1) + 1)(t + 1)$ because $x(x(t + 1) + 1)(t + 1) = (t(x + 1) + 1)((x + 1)x = 0$ by P_6.

Thus by Proposition 2.4, we have $x^2y \leq xy(1 + (x(t + 1) + 1)(t + 1))$. Since $t = z + 1$, consider

$$xy(1 + (x(t + 1) + 1)(t + 1))(xyt + 1)(x(t + 1) + 1) = xy(1 + (xz + 1)z)(xy(z + 1) + 1)(xz + 1).$$

By P_2 we have $xy(1 + (xz + 1)z)(xy(z + 1) + 1)(xz + 1) = xy(xz + 1)(1 + (xz + 1)z)(xy(z + 1) + 1)$.

By P_1, P_6 and P_2 , we have $xy(xz + 1)(1 + (xz + 1)z)(xy(z + 1) + 1) = xy(z + 1)(1 + (z + 1)xz)(xy(z + 1) + 1) = xy(z + 1)(xy(z + 1) + 1)(1 + (z + 1)xz) = 0$.

Hence,

$$yx^2 \leq xy \cdot (1 + (x(t + 1) + 1)(t + 1)) \leq 1 + (xyt + 1)(x(t + 1) + 1).$$

Thus by (iii) $xy \in I$.

(iv \Rightarrow i) is straight forward.

Corollary 2.1. Let I be an ideal of R . If I is implicative ideal, then $Nil(R) \subseteq I$ where $Nil(R)$ is set of all nil-potent elements of R .

Proposition 2.2. If $IM(R)$ is set of all implicative ideals of R , then a nil-radical

$$N \subseteq \bigcap_{I \in IM(R)} I.$$

Proof. Clearly $\bigcap_{I \in IM(R)} I$ is an implicative

ideal of R . It follows $Nil(R) \subseteq \bigcap_{I \in IM(R)} I$.

Thus, by Definition 1.5, $N \subseteq \bigcap_{I \in \text{IM}(R)} I$.

Theorem 2.2. Let I be an ideal of R . Then the following statements are equivalent.

1. I is implicative.
2. $\{x \in R : x^2 = 0\} \subseteq I$.
3. For all $x \in R, x(1 + x^2) \in I$.

Proof. (1 \Rightarrow 2). Let $y \in \{x \in R : x^2 = 0\} \Rightarrow y^2 = 0 \in I \Rightarrow 1 \cdot y^2 \in I$.

From proposition 2.1 (iii), we have $1 \cdot y = y \in I$.

(2 \Rightarrow 3). Let $x \in R$, consider $(x(1 + x^2))^2 = x^2(1 + x^2) \cdot (1 + x^2) = 0$. By (2) it follows that $x(1 + x^2) \in I, \forall x \in R$.

(3 \Rightarrow 1). Let $xy^2 \in I$. Since $x^2y^2 \leq xy^2 \in I \Rightarrow (xy)^2 \in I$. By (3) it follows that $xy(1 + (xy)^2) \in I$ and from Proposition 1.2, we have $xy \in I$. Therefore by Proposition 2.1, I is implicative.

Corollary 2.2. Let I be an implicative ideal of R and J be any ideal of R . If $I \subseteq J$, then J is an implicative ideal of R .

Corollary 2.3. If R has only zero as its nilpotent element, then $I = \{0\}$ is an implicative ideal of R .

Remark 2.1. If $K = \{x \in R : x^2 = 0\}$ is an ideal, then K is implicative.

Theorem 2.3. Let I be an ideal of R . I is a prime and an implicative ideal if and only if either $x \in I$ or $x + 1 \in I$.

Proof. Let $x \in R$. Since I is implicative by Proposition 2.2, $x(1 + ((x + 1) + 1)x) = x(1 + x^2) \in I$. As I is a prime ideal either $x((x + 1) + 1) = x^2 \in I$ or

$(x + 1)^2 \in I$. If $x^2 \in I$, from Theorem 2.1, we have $x \in I$. If $(x + 1)^2 \in I \Rightarrow x + 1 \in I$. Conversely suppose either $x \in I$ or $x + 1 \in I$ for any $x \in R$. Since I is an ideal and $x(1 + x^2) \leq x, x + 1 \Rightarrow x(1 + x^2) \in I$. By Theorem 2.2, I is implicative. Let $x, y \in R$. Since $x(y + 1) \leq x$ and $y(x + 1) \leq x + 1$, by assumption either $x(y + 1) \in I$ or $y(x + 1) \in I$. Thus I is prime.

Corollary 2.4. P is prime and implicative if and only if P is a maximal ideal.

Proof. According to Theorem 1.2, a proper ideal P of R is a maximal ideal if and only if, for every element x in R , either x is in P or $x + 1$ is in P . Therefore, by Theorem 2.3, we can deduce that P is both prime and implicative.

Proposition 2.3. Let I be the proper implicative ideal of R . Then I is prime if and only if I is the maximal ideal.

Proof. The forward proof follows from Corollary 2.4 and the converse proof is straightforward.

Theorem 2.4. For every P which is both an implicative and a prime ideal of R , there exists an atom h in the atomic pseudoring R such that $P = h^\perp$.

Proof. Let P be both an implicative and a prime ideal. By Corollary 2.4 and Theorem 1.2, the theorem holds.

Note that every ideal that is an implicative and prime ideal of a pseudoring is polar.

Proposition 2.4. Let $\varphi : R_1 \rightarrow R_2$ be a surjective homomorphism of a pseudoring. If I is an implicative ideal of R_1 and $\ker(\varphi) \subseteq I$, then $\varphi(I)$ is an implicative ideal of R_2 .

Proof. Let I be implicative ideal of R_1 and $\ker(\varphi) \subseteq I$. By Proposition 2.1, $\varphi(I)$ is an ideal of R_2 . Let $x, y, z \in R_2$ with $x \cdot y \cdot z \in \varphi(I)$ and $x(y + 1) \in \varphi(I)$. Since φ surjective there exists $a, b, c \in R_1$ such that $\varphi(a) = x, \varphi(b) = y, \varphi(c) = z, \Rightarrow \varphi(abc) \in \varphi(I)$ and $\varphi(a(b + 1)) \in \varphi(I) \Rightarrow abc \in I$ and $a(b + 1) \in I$. Since I is implicative $ac \in I \Rightarrow xz \in \varphi(I)$.

Proposition 2.5. If R is a Boolean ring, then each ideal I of R is implicative.

Proof. Let $xyz \in I$ with $x(y + 1) \in I$, for I is an ideal of R . It is clear that $xz(y + 1) \leq x(y + 1)$. By Definition 1.1 $xz(y + 1) \in I$. Since R is Boolean ring and by Proposition 1.1, $xz(1 + xz(y + 1)) = xyz \in I \Rightarrow xz \in I$. Therefore I is implicative.

Proposition 2.6. In Proposition 1.5, if I is an implicative ideal of R , then the kernel of φ_I or $(\ker(\varphi_I))$ is an implicative ideal of R .

Proof. Let $xyz \in \ker(\varphi_I)$ and $x(y + 1) \in \ker(\varphi_I) \Rightarrow \varphi_I(xyz) = \bar{0}$ and $\varphi_I(x(y + 1)) = \bar{0} \Rightarrow xyz \in I$ and $x(y + 1) \in I \Rightarrow xz \in I \Rightarrow xz \in \ker(\varphi_I)$.

Definition 2.2. Let $H \subseteq R$ is said to have the finite product property if $1 + (h_1 + 1)(h_2 + 1) \cdots (h_n + 1) \neq 1$ for any $h_1, \dots, h_n \in H$ and $h_i \neq 1$, for $1 \leq i \leq n$

Example 2.1. Let $R = \{0, a, b, 1\}$ be a set. R and the operations '+' and '.' is defined as:

+	0	a	b	1	.	0	a	b	1
0	0	a	b	1	0	0	0	0	0
a	a	0	1	b	a	0	a	0	a
b	b	1	0	a	b	0	0	b	b
1	1	b	a	0	1	0	a	b	1

Since we can easily see that $R = (R, +, \cdot, 1)$ is a pseudoring, $H = \{0, a\}$ has a finite product property, but $J = \{0, a, b\}$ has no finite product property because $1 + (a + 1)(b + 1) = 1$.

Example 2.2. Every singleton set different from the set containing unity in the Boolean ring has a finite product property.

Theorem 2.5. Let $H \subseteq R$ and $1 \notin H$. Then $\langle H \rangle$ is a proper ideal of R if and only if H has the finite product property.

Proof. Let $\langle H \rangle$ be a proper ideal of R . And assume that H has no finite product property. Then there are some of the elements in H , i.e. there exist $h_1, \dots, h_n \in H$ such that $1 + \prod_{i=1}^n (h_i + 1) = 1$ by Lemma 2.1, $1 \in \langle H \rangle$. Hence $\langle H \rangle = R$ implies contradiction to $\langle H \rangle$ is proper ideal of R .

Conversely, Suppose H has a finite product property. Assume that $\langle H \rangle = R \Rightarrow 1 \in \langle H \rangle$ by Lemma 2.1, $1 \leq 1 + \prod_{i=1}^n (h_i + 1)$ for some $h_i \in H$ which is a contradiction to H having a finite product property.

Proposition 2.7. Let $x \neq 1$ be in R and I be an ideal of R such that I has the finite product property. If $x \in I$ and $x + 1 \notin I$, then $I \cup \{x\}$ has the finite product property.

Proof. Let $B = I \cup \{x\}$. We need to show that $1 + (b_1 + 1)(b_2 + 1) \cdots (b_n + 1) \neq 1$ for any $b_1, b_2, \dots, b_n \in B$ and $b_i \neq 1$.

Assume $1 + (b_1 + 1)(b_2 + 1) \cdots (b_n + 1) = 1$

for some $b_1, b_2, \dots, b_n \in B$. If $b_1, b_2, \dots, b_n \in I$, then it is clear that by Definition 1.1, $1 \in I$. This contradicts the fact that I has finite product property.

Without loss of generality let $b_1 = x$ and $1 + (b_1 + 1)(b_2 + 1) \dots (b_n + 1) = 1$

for some $b_1, b_2, \dots, b_n \in B \Rightarrow 1 + (x + 1)((1 + (b_2 + 1)(b_3 + 1) \dots (b_n + 1)) + 1) = 1 + (x + 1)(b_2 + 1)(b_3 + 1) \dots (b_n + 1) = 1$

implies $(x + 1)(b_2 + 1)(b_3 + 1) \dots (b_n + 1) = 0 \in I$.

Since $1 + (b_2 + 1)(b_3 + 1) \dots (b_n + 1) \in I$ by Proposition 1.1, we have $x + 1 \in I$ which is a contradiction. Therefore in both case $1 + (b_1 + 1)(b_2 + 1)(b_3 + 1) \dots (b_n + 1) \neq 1$

for any $b_1, b_2, \dots, b_n \in B$. and so $I \cup \{x\}$ has the finite product property.

Proposition 2.8. Let $\varphi : R_1 \rightarrow R_2$ be a homomorphism, then the following holds;

- i. If $H \subseteq R_1$ has a finite product property and φ is one to one, then $\varphi(H)$ has a finite product property.
- ii. If $H \subseteq R_2$ has a finite product property and φ is onto, then $\varphi^{-1}(H)$ has a finite product property.

Proof

i. Let $t_1, t_2, \dots, t_n \in \varphi(H)$ such that $(t_1 + 1)(t_2 + 1) \dots (t_n + 1) = 0$. It follows that there are $h_1, h_2, \dots, h_n \in H$ such that $\varphi(h_1) = t_1, \varphi(h_2) = t_2, \dots, \varphi(h_n) = t_n \Rightarrow \varphi((h_1 + 1)(h_2 + 1) \dots (h_n + 1)) = 0 \Rightarrow (h_1 + 1)(h_2 + 1) \dots (h_n + 1) \in \ker(\varphi) = \{0\}$. This is a contradiction. Hence $\varphi(H)$ has a finite product property.

ii. Assume that there exists $t_1, t_2, \dots, t_n \in \varphi^{-1}(H)$ with $(t_1 + 1)(t_2 + 1) \dots (t_n + 1) = 0 \Rightarrow \varphi(h_1 + 1)(h_2 + 1) \dots (h_n + 1) = 0$ for some $h_i \in H$ and $\varphi(h_i) = t_i$. This is a contradiction that H has finite product property.

Proposition 2.9. Let $At(R)$ be the set of all atoms in R . Then $At(R)$ has a finite product property.

Proof. Assume there exist $h_i \in At(R)$, for $i = 1, 2, \dots, n$ and for each $h_i \neq 1$ with $1 + (h_1 + 1)(h_2 + 1) \dots (h_n + 1) = 1 \Rightarrow (h_1 + 1)(h_2 + 1) \dots (h_n + 1) = 0$

By Definition 4.1.1, either $(h_2 + 1) \dots (h_n + 1) = h_1$ or $(h_2 + 1) \dots (h_n + 1) = 0$.

Case 1: If $(h_2 + 1) \dots (h_n + 1) = h_1$, then $h_1 (h_2 + 1) \dots (h_n + 1) = 0$

$$\Rightarrow h_1(h_3 + 1) \dots (h_n + 1) \leq h_2 \Rightarrow h_1(h_3 + 1) \dots (h_n + 1) = h_2$$

or

$h_1(h_3 + 1) \dots (h_n + 1) = 0$. If $h_1(h_3 + 1) \dots (h_n + 1) = h_2 \Rightarrow h_2 \leq h_1$. Since h_i 's are distinct, $h_2 = 0$. This contradicts that h_2 is an atom. Thus $h_1(h_3 + 1) \dots (h_n + 1) = 0$. It follows that $h_1(h_4 + 1) \dots (h_n + 1) = h_3$ or $h_1(h_4 + 1) \dots (h_n + 1) = 0$. Similarly, if $h_1(h_4 + 1) \dots (h_n + 1) = h_3 \Rightarrow h_3 = 0$ and this contradicts h_3 is an atom. Thus $h_1(h_4 + 1) \dots (h_n + 1) = 0$. Proceeding like this we get $h_1(h_n + 1) = 0 \Rightarrow h_1 = 0$. This is also contradiction. Therefore $h_1 \neq (h_2 + 1) \dots (h_n + 1)$.

Case 2: If $(h_2 + 1) \dots (h_n + 1) = 0$, then by Definition 4.1.1 and mathematical induction, $h_i = 1$ for some i . This is a contradiction.

Hence $1 + \prod_{i=1}^n (h_i + 1) \neq 1$.

Proposition 2.10. If I is a proper ideal of R , then I has a finite product property.

Proof: Assume $1 + \prod_{i=1}^n (h_i + 1) = 1$, for some elements $h_i \in I$. It follows from Definition 1.1, $1 \in I \Rightarrow I = R$ which is a contradiction.

Thus, from Corollary 2.4 and Proposition 2.10, we conclude that if I is an implicative and prime ideal of R , then I has a finite product property.

Theorem 2.6. If I is a proper implicative ideal of R , then $I^{\perp\perp}$ has a finite product property.

Proof. Let I be an implicative ideal properly contained in R . Since from Proposition 2.10, it follows that I has a finite product property and by Proposition 1.5, $I \subseteq I^{\perp\perp}$, by Proposition 1.6, we conclude that $I^{\perp\perp}$ is a properly contained in R . From Theorem 1.4, it follows that $I^{\perp\perp}$ is an ideal of R . Consequently, by Proposition 2.10, the theorem holds.

Conclusion

To summarize, an implicative ideal is a distinct sort of ideal in a pseudoring with unique qualities. Implicative ideals are also important for describing the algebraic structure of an ideal and its relationship to other types of ideals. Implicative ideal of a pseudoring, like it does in other algebraic structures, has applications in fuzzy logic. In this study, the authors define the notion of implicative ideal of a pseudoring and various theorems, propositions, corollaries, and lemmas were given, and their proofs were demonstrated. The researchers also demonstrated that every prime ideal is implicative if and only if it is a maximum ideal. Furthermore, the finite product property of a subset of a pseudoring is defined and some of its characteristics are proven. It is useful for scholars who need to investigate the structure and properties of ideals of a pseudoring in depth. The relationship between a subset of a

pseudoring with the finite product property and the implicative ideal was proved. Finally, it was shown that an implicative ideal has a relationship with prime, polar ideals, and the atom of an atomic pseudoring.

Contribution statement

The first author is the one who came up with the concept of finite product properties. He also establishes the relationship between the ideal that is generated by this subset of a pseudoring and the other sorts of ideals, particularly the implicative ideal of a pseudoring. Furthermore, all theorems and assertions are originated and proven by the first author.

The second author was the one who came up with the concept of the implicative ideal in the case of pseudoring. He also served as a guide during the entire process of writing the document, particularly with regard to the verification of all the features.

The third author provided a critical evaluation and revised the article.

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Declaration

The authors declare that the manuscript is original, has never been published, and is not currently under consideration for publication elsewhere and no data sets were generated or analyzed during the current study

Conflict of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper

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